Chapter 1 – Logical Thinking

Part I – Inductive and Deductive Reasoning

We will begin this section with two definitions.

Inductive reasoning is a type of reasoning where a person arrives at a conclusion by looking at several examples and noticing a pattern.

Deductive reasoning is a type of reasoning where a person arrives at a conclusion by realizing that it is a special case of a general principle that is known to be true.

Let's look at some examples that illustrate these definitions.

Example 1: You know that all fish have gills. You see a catfish, and you decide that it must have gills. Is this an example of inductive or deductive reasoning?

This is an example of deductive reasoning. You know that the general principle that all fish have gills is true. You also know that a catfish is a specific type of fish, and so you apply the general principle that all fish must have gills to this specific case.

Example 2: A two-year-old boy and his mother are walking through a park. The child's mother sees a hummingbird. She points to it and says, "bird." When they see a pigeon later on, the mother points to it and says "bird" again. A little bit later, a fly lands on the child's hand, and the child says "bird." Is this an example of inductive or deductive reasoning?

This is an example of inductive reasoning. The boy noticed that all the birds he saw had wings and could fly. The child used this information to conclude that, since the fly had wings and could fly, it had to be a bird also.

You should realize that you will always reach a correct conclusion using deductive reasoning *if you use it correctly*. When you use inductive reasoning, you will reach a correct conclusion some of the time, but not always (as shown by the last example).

Example 3: Find the next two numbers in the sequence 5, 7, 9, 11, 13, …. Did you use inductive or deductive reasoning?

If you look at the first couple of numbers in the sequence, you should see that, in order to get each successive number, you add two to the previous number. This tells us that the next two numbers in this sequence are probably 15 and 17. This is an example of inductive reasoning because we are looking at the first five numbers in the sequence and noticing a pattern. Now, notice that we said that the next two numbers are *probably* 15 and 17. This is because we do not know for sure that the next two numbers are 15 and 17. For instance, the sequence could start back over at 5 after the 13 (which would mean that the sequence is really 5, 7, 9, 11, 13, 5, 7, 9, 11, 13, 5, 7, 9, 11, 13, …).

Since inductive reasoning can be wrong, it has no place in our discussion of logical reasoning. However, both inductive reasoning and deductive reasoning are important in mathematics. Many mathematical ideas have been discovered because a mathematician looked at several examples and noticed a pattern. For example, you may know that the measures of the angles in a triangle add up to 180º. However, you would probably never figure this out by starting with the properties of a triangle and aimlessly using deductive reasoning. You would be much more likely to discover this by studying lots of triangles and noticing the pattern.

Now, we need another definition.

A <u>conjecture</u> is a conclusion that a person reaches using inductive reasoning.

So, for example, we could say that, in Example 2, the two-year-old child made a conjecture that the fly that landed on his hand was a bird. In Example 3, we made a conjecture that the next two numbers in the sequence were 15 and 17.

Example 4: Mark draws five quadrilaterals (polygons with four sides) on his paper. He notices that, in each figure, the measures of the angles add up to 360º. What conjecture might he make based on this observation?

Mark might make the conjecture that the angles of any quadrilateral must add up to 360º.

Problems:

1-10. State whether inductive or deductive reasoning was used to reach each conclusion.

1. Stacey knows that anyone who lives in Orlando lives in Florida. She also knows that Jacqueline lives in Orlando, and so she concludes that Jacqueline must live in Florida.

2. Fido the dog has noticed that, every time his owner pulls out a leash, he is about to go for a walk. Fido sees his owner reaching for his leash, and so he concludes that he is about to go for a walk.

3. Bonnie has noticed that, every time she feeds her daughter peanuts, her daughter breaks out in hives. Bonnie concludes that her daughter must be allergic to peanuts.

4. Robert knows that his mother has fixed spaghetti for dinner every Saturday night for the past month. He also knows that today is Friday, and so he concludes that his mother will fix spaghetti for dinner tomorrow night.

5. Janet knows that all new patients of Dr. Taylor must fill out an information form and that she has her first appointment with him at 11:00 a.m. today. She concludes that she will have to fill out an information form when she goes in for her appointment.

6. Every time that Nadine has tried to order a certain camera from an online electronics store, the store has not had any in stock. She decides that, when she tries to order the camera from the store later today, the camera will probably be out of stock.

7. Helen knows that all odd numbers are not divisible by 2 and that 457 is an odd number. She concludes that 457 is not divisible by 2.

8. Chad wants to buy a chair. He sees a sign outside a furniture store that says, "All Chairs on Sale." When he finds a chair he likes, he concludes that the chair he wants to buy must be on sale.

9. Lori's Spanish teacher points to a blue spot on the wall and says, "*azul*." She points to a blue book and says "*azul*" again. Lori concludes that the word "*azul*" must mean blue in Spanish.

10. Jack's teacher tells him that perpendicular lines form right angles when they intersect. He also knows that line m and line n are perpendicular, and so he concludes that line m and line n must form right angles when they intersect.

11-20. Make a conjecture based on the information given to you.

11. The first six numbers of a sequence are 12, 19, 26, 33, 40, and 47.

    Conjecture: The next two numbers of the sequence are _______ and _______.

12. The first six numbers of a sequence are 1, –1, 1, –1, 1, and –1.

    Conjecture: The next two numbers of the sequence are _______ and _______.

13. 7 + 2 = 9 | 51 + 42 = 93 | 33 + 54 = 87

5 + 12 = 17 | 75 + 10 = 85 | 67 + 84 = 151

Conjecture: The sum of an odd number and an even number is an _________ number.

14. 5 + 0 = 5 | –17 + 0 = –17 | –62 + 0 = –62

12 + 0 = 12 | 43 + 0 = 43 | –158 + 0 = –158

Conjecture: ____

____

15. Adam has noticed that, for the past six weeks, the cafeteria at the college he is attending has served hamburgers every Friday. He also knows that today is Friday and that the cafeteria will be open today.

Conjecture: ____

____

16. Vanessa has noticed that it rains every day in the summer in the town she has just moved to. Today is a warm summer day.

Conjecture: ____

____

17. Suppose that you know that your local newspaper has correctly picked the winner of the election for your town's mayor for each of the past eighteen elections. Also suppose that the same newspaper has picked Pat Green to win the next election for the mayor of your town.

Conjecture: ____

____

18. For the past two weeks, Martha has heard a dog barking whenever she walks by the house at 123 Main Street, and she is about to walk by the house at 123 Main Street.

Conjecture: ____

____

19. Simon has noticed that, whenever he watches his favorite football team on television, they always lose. He knows that his favorite football team is playing at 7:00 tonight, and he plans to watch the game on television.

    Conjecture: ____

    ____

20. Carrie has just started a new job, and her boss has told paychecks will be issued every Friday. Today is Thursday.

    Conjecture: ____

    ____

21-22. Test some numbers and make a conjecture about each of the following.

21. The product of an odd number and an even number is an _________ number.

22. If a number is divisible by both 4 and 7, then it is also divisible by which of the following? (Circle all that apply.)

    (a) 2 (b) 14 (c) 6 (d) 28 (e) 56

Part II – Evaluating the Validity of Statements and Arguments

In this section, we will discuss how to tell if statements and arguments are true or false. Before we look some examples, we will note that statements that begin with the word "all" can often be rewritten as a statement with the word "if" at the beginning and the word "then" in the middle. For instance, the statement, "All triangles have three sides" can be rewritten as, "If a figure is a triangle, then it has three sides."

Example 1: Decide whether the following statement is true or false: *All squares are rectangles*.

We will begin by noting that a square is defined to be a polygon with four sides that have the same length and four right angles, and a rectangle is defined to be a polygon with four sides and four right angles. We will also note that this statement can be rewritten as, "If a figure is a square, then it is a rectangle." To figure out if this statement is true or false, we must start by thinking about a square – a polygon with four sides that have the same length and four right angles. Since this figure must have four sides and four right angles, we can say that our figure must also be a rectangle. This means that the statement, "All squares are rectangles" is true.

Example 2: Decide whether the following statement is true or false: *Some rectangles are squares.*

This question wants us to start by thinking about a rectangle, which is a figure with four sides and four right angles. Since this figure could have four sides that all have the same length, this figure could be a square. Hence, this statement is true.

Example 3: Suppose that a certain restaurant offers a buffet, and the prices for different age groups are listed at the right. In one of their advertisements, they say, "If you are a senior citizen, then your price to eat at the buffet is only $5.99." Under what circumstances would this statement be false?

Buffet Prices	
Children (12 years old and younger)	$5.99
Adults (ages 13-54)	$6.99
Senior Citizens (55 years and older)	$5.99

We will consider four different situations: (a) You are a senior citizen and you are charged $5.99 to eat at the buffet, (b) You are a senior citizen and your price to eat at the buffet is not $5.99, (c) You are not a senior citizen and you are charged $5.99 to eat at the buffet, and (d) You are not a senior citizen and your price to eat at the buffet is not $5.99.

(a) You are a senior citizen and your price to eat at the buffet is $5.99.

If this happens, then the advertisement is true.

(b) You are a senior citizen and your price to eat at the buffet is not $5.99.

If this happens, then the advertisement is false.

(c) You are not a senior citizen and your price to eat at the buffet is $5.99.

If this happens, then the advertisement is true. Notice that the advertisement makes no claim about what happens if you are not a senior citizen.

(d) You are not a senior citizen and your price to eat at the buffet is not $5.99.

If this happens, then the advertisement is true. Again, notice that the advertisement makes no claim about what happens if you are not a senior citizen.

This tells us that the only time this advertisement would be false is when a person is a senior citizen and is charged more or less than $5.99 to eat at the buffet.

This example gives rise to the following rule: Whenever the "if" part of a statement is false, it does not matter what the "then" part says – the original statement is true. Also, the only way that a statement in the form "if …, then …" can be false is if the "if" part of the statement is true and the "then" part of the statement is false.

Example 4: Tell whether the following statement is true or false: *If 5 = –2, then 23 is divisible by 2*.

Since the "if" part of this statement is false, we can say that the entire statement must be true. The number 23 is obviously not divisible by 2, but this statement makes no claim about what happens when 5 is not equal to –2.

Now, we need two rules. The first is called the <u>Law of Detachment</u>, and the second is called the <u>Law of Syllogism</u>.

Let *p* and *q* represent two statements, and suppose that you know that the statement, "If *p*, then *q*" is true. Also suppose that you know that *p* is true. Then *q* must also be true.

Let *p*, *q*, and *r* represent three statements, and suppose that you know that the statements, "If *p*, then *q*" and "If *q*, then *r*" are both true. Then the statement, "If *p*, then *r*" is also true.

Example 5: Assume that the following statements are true: (a) If a figure is a rectangle, then its diagonals are congruent. (b) Polygon ABCD is a rectangle. Can you reach a conclusion that must be true based on these statements?

We can let *p* represent the statement, "A figure is a rectangle," and we can let *q* represent the statement, "The diagonals of the figure are congruent." This gives us a statement in the form "If *p*, then *q*" that we know is true. Also, we know that the statement *p* is true for polygon ABCD. Therefore, we can use the Law of Detachment to conclude that the statement *q* is also true for polygon ABCD. This tells us that we can say that the diagonals of polygon ABCD are congruent.

Example 6: Assume that the following statements are true: (a) If an animal is a rabbit, then it has fur. (b) Buster is not a rabbit. Can you reach a conclusion that must be true based on these statements?

We will let *p* represent the statement, "An animal is a rabbit," and we will let *q* represent the statement, "It has fur." This gives us a statement in the form "If *p*, then *q*" that we know is true. We also know that the statement *p* is not true for Buster. Note that neither the Law of Detachment nor the Law of Syllogism talks about what happens when *p* is not true, and so we cannot reach a conclusion that must be true based on these statements.

Example 7: Assume that the following statements are true: (a) If a figure is a rectangle, then its diagonals are congruent. (b) All squares are rectangles. Can you reach a conclusion that must be true based on these statements?

We will begin by noting that (b) can be rewritten as, "If a figure is a square, then it is a rectangle." Now, we will let *p* represent the statement, "A figure is a square," we will let *q* represent the statement, "A figure is a rectangle," and we will let *r* represent the statement, "The diagonals of the figure are congruent." This gives us a statement in the form "If *p*, then *q*" and a statement in the form, "If *q*, then *r*," as shown below.

If *p*, then *q*.
If a figure is a square, then it is a rectangle.

If *q*, then *r*.
If a figure is a rectangle, then its diagonals are congruent.

Since we know that both our "If *p*, then *q*" and our "If *q*, then *r*" statements are true, we can use the Law of Syllogism to conclude that "If *p*, then *r*" must also be true. This tells us that we can say that the statement, "If a figure is a square, then its diagonals are congruent" is true.

Example 8: Assume that the following statements are true: (a) If a figure is a square, then it is a rectangle. (b) If a figure is a square, then it is a rhombus. Can you reach a conclusion that must be true based on these statements?

We will let *p* represent the statement, "A figure is a square," we will let *q* represent the statement, "A figure is a rectangle," and we will let *r* represent the statement, "A figure is a rhombus." This gives us a statement in the form "If *p*, then *q*" and a statement in the form, "If *p*, then *r*," as shown below.

If *p*, then *q*.
If a figure is a square, then it is a rectangle.

If *p*, then *r*.
If a figure is a square, then it is a rhombus.

Neither the Law of Detachment nor the Law of Syllogism makes any claim about what happens when we know that two statements of the form "If *p*, then *q*," and "If *p*, then *r*," are true, and so we cannot make a conclusion from these two statements.

Now, we need one more definition.

A <u>counterexample</u> is an example that shows that a conjecture is false.

Example 9: Find a counterexample to the statement, "All mammals live on land."

We will begin by noting that this statement can be rewritten as, "If an animal is a mammal, then it lives on land." Now, to find a counterexample, we must find an example that proves that this statement is false. As we said after Example 3, the only way that a statement in the form "if …, then …" can be false is if the "if" part of the statement is true *and* the "then" part of the statement is false. So, in order to prove this statement false, we must find an example of an animal that is a mammal and does not live on land. Since a dolphin is a mammal that does not live on land, we can say that a dolphin is a counterexample to this statement.

Problems:

1-20. Decide whether each of following statements is *true* or *false*.

1. If $x = 8$, then $x – 5 = 3$.

2. If $x^2 = 9$, then $x = 3$.

3. If $x^2 = 9$, then x could equal 3.

4. Some triangles have three sides.

5. If $4 < 1$, then $5 = 2$.

6. If $3 = 4$, then $0 < 1$.

7. Some dogs have spots.

8. All unicorns on Saturn love pepperoni pizza.

9. There is a unicorn on Saturn that loves pepperoni pizza.

10. All numbers that are greater than 7 are greater than 10.

11. All numbers that are greater than 10 are greater than 7.

12. Some people who live in Chicago live in Illinois.

13. All people who live in Chicago live in Illinois.

14. All people who live in Illinois live in Chicago.

15. If $a = 10$, then $a – 4 = 6$.

16. If $7 < 8$, then pigs can fly.

17. If pigs can fly, then $7 < 8$.

18. If pigs can fly, then $7 > 8$.

19. All rectangles have four sides.

20. All figures with four sides are rectangles.

21-30. For each statement below, state or draw a counterexample that shows that the statement is false.

21. All prime numbers are odd.

22. All odd numbers are prime.

23. If x is a real number, then $x > \frac{1}{x}$.

24. If x is a real number, then $x^2 > x$.

25. All rational numbers are whole numbers. (Recall that rational numbers are numbers that can be written as either fractions or as decimals that either stop or repeat, and the whole numbers are 0, 1, 2, 3, 4, 5, 6, ….)

26. If a figure has four sides, then it is a square.

27. If a figure has four sides, then it is <u>not</u> a square.

28. If $x > 3$, then $x^2 > 15$.

29. If a number is divisible by 6, then it is also divisible by 12.

30. If a number is divisible by 6, then it is <u>not</u> divisible by 12.

31-42. Assume that each of the statements below are true, and then use either the Law of Detachment or the Law of Syllogism to write a conclusion that must be true *if possible*.

31. (a) If today is a holiday, then Calvin will not go to school today.
    (b) Today is not a holiday.

32. (a) If you do your homework, then you will get better grades.
    (b) If you get better grades, then you will get into a better college.

33. (a) If a figure is a rhombus, then it is a quadrilateral.
    (b) If a figure is a square, then it is a rhombus.

34. (a) If an animal is a bird, then it has feathers.
    (b) A cardinal is a type of bird.

35. (a) A number is said to be rational if it can be written as the ratio of two integers.
    (b) The number 3 can be written as the ratio $\frac{3}{1}$.

36. (a) If Tammy has school tomorrow, then she must be home by 8:30 p.m.
    (b) Tammy must be home by 8:30 p.m. tonight.

37. (a) If it is before 9 p.m., then the pharmacy is open.
    (b) If the pharmacy is open, then Helen will get her medicine.

38. (a) If a number is a natural number, then it is a whole number.
    (b) All whole numbers are integers.

39. (a) Anyone who lives in Alabama lives in North America.
    (b) Anyone who lives in Arkansas lives in North America.

40. (a) If $y = 3$, then $x = 4$.
    (b) $x = 4$

41. (a) If point B is between points A and C, then $AB + BC = AC$.
    (b) $AB + BC = AC$.

42. (a) If a number is less than 4, then it is less than 7.
    (b) If a number is less than 4, then it is less than 12.

Part III – The Converse, Inverse, and Contrapositive of Statements

We will begin this section with a discussion of a concept called the negation of statements.

When you negate a statement, you say that the statement is false.

When you negate a statement correctly, either the original statement or the negation of the statement must be true. So, for example, we could not negate the statement, “The apple is red” by saying, “The apple is green” because this leaves out the possibility that the apple is yellow, brown, or some other color.

Example 1: Consider the following statement: *Everyone likes Sylvester*. Which of the following correctly negates this statement? (a) There is at least one person somewhere who does not like Sylvester. (b) No one likes Sylvester. (c) Everyone dislikes Sylvester. (d) It is false that everyone likes Sylvester.

To negate this statement, we must figure out how we can say, “It is not true that everyone likes Sylvester.”

(a) This is a correct way of negating the original statement. If we look at the statements, “Everyone likes Sylvester” and “There is at least one person somewhere who does not like Sylvester,” we see that we are not leaving out any possibilities.

(b) This is not a correct way of negating the original statement. If we look at the statements, “Everyone likes Sylvester” and “No one likes Sylvester,” we see that these two statements leave out the possibility that some people like Sylvester and some people don’t.

(c) This statement is equivalent to statement (b), and so it is not a correct way of negating the original statement.

(d) This is a correct way of negating the original statement. If we look at the statements, “Everyone likes Sylvester” and “It is false that everyone likes Sylvester,” we see that we are not leaving out any possibilities.

Now, we need three more definitions.

The converse of a statement is formed by switching the “if” part and the “then” part.

The inverse of a statement is formed by negating both the “if” part and the “then” part.

The contrapositive of a statement is formed by switching the “if” part and the “then” part and negating both parts.

In the next two examples, we list two ways of writing the converse, inverse, and contrapositive of the statements, but these are not the only correct answers. There are many other ways to write these statements.

Example 2: Consider the following statement: *If a number is divisible by 4, then it is divisible by 3*. Write the converse, inverse, and contrapositive of this statement, and tell whether each is true or false.

We will begin by noting that, because the number 8 serves as a counterexample to the original statement, we can say that the original statement is false.

The converse of this statement can be written as *If a number is divisible by 3, then it is divisible by 4* or as *A number is divisible by 4 if it is divisible by 3*. The number 9 is a counterexample to either of these forms of the converse, and so we can say that the converse of this statement is false.

The inverse of this statement can be written as *If a number is not divisible by 4, then it is not divisible by 3* or as *A number is not divisible by 3 if it is not divisible by 4*. The number 9 is a counterexample to either of these forms of the inverse, and so we can say that the inverse of this statement is false.

The contrapositive of this statement can be written as *If a number is not divisible by 3, then it is not divisible by 4* or as *A number is not divisible by 4 if it is not divisible by 3*. The number 8 is a counterexample to either of these forms of the contrapositive, and so we can say that the contrapositive of the statement is false.

Example 3: Consider the following statement: *All numbers that are less than 5 are less than 8*. Write the converse, inverse, and contrapositive of this statement, and tell whether each is true or false.

We will begin by noting that this statement can be rewritten as *If a number is less than 5, then it is less than 8*. We will also note that this statement is true.

To find the converse of this statement, we must switch the "if" part of the statement with the "then" part of the statement. This tells us that the converse of this statement can be written as *If a number is less than 8, then it is less than 5* or as *All numbers that are less than 8 are also less than 5*. Since the number 7 serves as a counterexample to both of these forms of the converse, we can say that the converse is false.

To find the inverse of this statement, we must negate both the "if" part and the "then" part. This tells us that we can write the inverse of the statement as *If a number is not less than 5, then it is not less than 8* or as *If a number is greater than or equal to 5, then it is greater than or equal to 8*. Since the number 5 serves as a counterexample to both of these forms of the inverse, we can say that inverse is false.

To find the contrapositive of this statement, we must switch the "if" part with the "then" part and negate both of them. This tells us that we can write the contrapositive of this statement as *If a number is not less than 8, then it is not less than 5* or as *All numbers that are greater than or equal to 8 are also greater than or equal to 5*. Both of these statements are true. (To see why, think of a number that is greater than or equal to 8. This number must be greater than or equal to 5.)

Finally, we will note that, if the original statement is true, then the contrapositive of the statement must be true (and, if the original statement is false, then the contrapositive of the statement must be false). If the original statement is true, then the converse and inverse of the statement may or may not be true.

Example 4: Assume that the following statement is true: *All kowsers are green*. Which of the following statements **must** be true? (a) If an object is a kowser, then it must be green. (b) If an object is green, then it is a kowser. (c) If an object is green, then it could be a kowser. (d) If an object is not green, then it is not a kowser. (e) All green objects are kowsers. (f) If an object is not a kowser, then it is not green. (g) Kowsers exist. (h) Kowsers do not exist.

(a) We can use what we discussed in the last section to say that the original statement can be rewritten to say, "If an object is a kowser, then it is green." Thus, statement (a) must be true.

(b) This is the converse of the original statement, and we have no way of knowing whether it is true or not.

(c) The original statement tells us that, if we ever find a kowser, we know that it will be green. So, if we find a green object somewhere, it could be a kowser. Therefore, this statement is true.

(d) This is the contrapositive of the original statement. Since we were told that the original statement was true, we know that this statement must also be true.

(e) This statement is saying the same thing as statement (b), and so we have no way of knowing whether or not this statement is true.

(f) This is the inverse of the original statement, and we have no way of knowing whether it is true or not.

(g) The original statement tells us that, if we ever find a kowser, we know that it will be green. It does not tell us that kowsers exist, and it does not tell us that kowsers do not exist. Thus, we have no way of knowing whether or not this statement is true.*

(h) As we said in (g), the original statement tells us that, if we ever find a kowser, we know that it will be green. It does not tell us that kowsers exist, and it does not tell us that kowsers do not exist. Thus, we have no way of knowing whether or not this statement is true.*

This tells us that the only statements that must be true are (a), (c), and (d).

*If (g) and (h) confuse you, think about this statement: *All unicorns have one horn*. This is a true statement because anyone drawing a unicorn will draw the unicorn with one horn, but of course unicorns do not really exist.

Problems:

1-10. For each of the given statements, select all of the answer choices that negate the original statement correctly.

1. Benjamin has a dog.
   (a) Benjamin does not have a dog.
   (b) Benjamin has a cat.
   (c) It is not true that Benjamin has a dog.
   (d) Gavin has a dog.
   (e) Gavin does not have a dog.

2. $m \leq n$
   (a) $m > n$
   (b) It is false that $m \leq n$.
   (c) $m < n$
   (d) $m = n$
   (e) $m \neq n$
   (f) $m \geq n$

3. I am not going to the movies.
   (a) I am going shopping.
   (b) I am going to the movies.
   (c) Sam is not going to the movies.
   (d) Sam is going to the movies.

4. The light is bright.
   (a) The light is not bright.
   (b) The light is dim.
   (c) The light is off.
   (d) It is not true that the light is bright.
   (e) The sun is bright.

5. "I do not like green eggs and ham." –*Green Eggs and Ham* by Dr. Seuss
   (a) I like green eggs and ham.
   (b) I like yellow eggs and ham.
   (c) I do not like yellow eggs and ham.

6. This bell pepper is green.
    (a) This bell pepper is red.
    (b) This bell pepper is blue.
    (c) It is false to say that this bell pepper is green.
    (d) This bell pepper is not green.
    (e) This cucumber is green.

7. $x < 5$
    (a) $x = 8$
    (b) $x > 5$
    (c) $x \geq 6$
    (d) $x \geq 5$
    (e) x is not less than 5

8. The glass is half full.
    (a) The glass is half empty.
    (b) The glass is empty.
    (c) The glass is full.
    (d) The glass is not half full.
    (e) The glass is ¼ full.

9. Some taggles are not pink.
    (a) There exists a taggle that is pink.
    (b) Some taggles are red.
    (c) Some taggles are not red.
    (d) All taggles are pink.

10. All dogs are cute.
    (a) All dogs are ugly.
    (b) All cats are ugly.
    (c) All cats are cute.
    (d) There is at least one dog somewhere that is not cute.
    (e) Some dogs are cute.
    (f) It is false that all dogs are cute.

11-18. Write the converse, inverse, and contrapositive of each of the following statements, and then tell whether each is true or false.

11. If $x = 8$, then $x^2 - 7 \neq 9$.

    True or false?

    converse:

    True or false?

    inverse:

    True or false?

    contrapositive:

    True or false?

12. If $x = 3$, then $x > 5$.

    True or false?

    converse:

    True or false?

    inverse:

    True or false?

    contrapositive:

    True or false?

13. All positive numbers are natural numbers.

    (Recall that the natural numbers are the numbers 1, 2, 3, 4, 5, 6, … . They do not include fractions, decimals, negative numbers, or zero.)

    True or false?

    converse:

    True or false?

    inverse:

    True or false?

    contrapositive:

    True or false?

14. If $3 \cdot 4 = 25$, then $5 + 1 = 2$.

True or false?

converse:

True or false?

inverse:

True or false?

contrapositive:

True or false?

15. If $x = 7$, then $2x = 14$.

True or false?

converse:

True or false?

inverse:

True or false?

contrapositive:

True or false?

16. All squares have four sides.

True or false?

converse:

True or false?

inverse:

True or false?

contrapositive:

True or false?

17. All triangles have 12 sides.

    True or false?

    converse:

    True or false?

    inverse:

    True or false?

    contrapositive:

    True or false?

18. All numbers that are divisible by 2 are divisible by 4.

    True or false?

    converse:

    True or false?

    inverse:

    True or false?

    contrapositive:

    True or false?

19. Write the converse, inverse, and contrapositive of the following statement: *Call me at 1-800-555-1234 if you are interested in buying a house*.

    converse:

    inverse:

    contrapositive:

20. Write the converse, inverse, and contrapositive of the following statement, originally said by Mark Twain: *If you tell the truth, you don't have to remember anything*.

    converse:

    inverse:

    contrapositive:

21. Assume that the following statement is true: *All customers should exit the theater through the rear doors*. Select all of the following statements that must be true.
    (a) If you are a customer, then you should exit the theater through one of the rear doors.
    (b) If you are not a customer, then you should exit the theater through one of the rear doors.
    (c) If you are not a customer, then you could exit the theater through one of the rear doors.
    (d) If you work at the theater, then you should not exit the theater through one of the rear doors.
    (e) If you work at the theater, then you should exit the theater through one of the rear doors.
    (f) If you have to exit through the rear doors, then you must be a customer.
    (g) If you do not have to exit through the rear doors, then you are not a customer.

22. Assume that the following statement is true: *All pacadams are brackle.* Select all of the following statements that must be true.
    (a) If something is brackle, then it is a pacadam.
    (b) If something is not brackle, then it is not a pacadam.
    (c) There exists a pacadam somewhere that is brackle.
    (d) If something is a pacadam, then it is brackle.
    (e) If something is brackle, then it could be a pacadam.
    (f) If something is not a pacadam, then it is not brackle.
    (g) If something is not a pacadam, then it could be brackle.

23. Assume that the following statement is true: *If Victoria is wearing her black pants, then she is wearing her black shoes.* Select all of the statements below that must be true.
    (a) If Victoria is wearing her black shoes, then she is wearing her black pants.
    (b) If Victoria is not wearing her black shoes, then she is not wearing her black pants.
    (c) If Victoria is not wearing her black pants, then she is not wearing her black shoes.
    (d) Victoria is wearing her black pants.
    (e) Victoria is not wearing her black pants.
    (f) Victoria is wearing her black shoes.
    (g) Victoria is not wearing her black shoes.

24. Assume that the following statement is true: *If Samantha apologizes for her actions, then Allen will apologize for his actions*. Select all of the statements below that must be true.

    (a) If Allen apologizes for his actions, then Samantha apologized for her actions.

    (b) If Samantha does not apologize for her actions, then Allen will not apologize for his actions.

    (c) If Allen does not apologize for his actions, then Samantha did not apologize for her actions.

    (d) Allen will apologize for his actions.

    (e) Allen will not apologize for his actions.

25. Assume that the following statement is true: *If an object is queely, then it is either yellow or brown.* Select all of the statements below that must be true.

    (a) If an object is not queely, then it is not yellow or brown.

    (b) All objects that are queely are either yellow or brown.

    (c) If an object is not yellow or brown, then it is not queely.

    (d) If an object is either yellow or brown, then it is queely.

26. Assume that the following statement is true: *All molecules of water are made of hydrogen and oxygen*. Select all of the statements below that must be true.

    (a) If it is not a molecule of water, then it is not made of hydrogen and oxygen.

    (b) If it is not made of hydrogen and oxygen, then it is not a molecule of water.

    (c) If it is a molecule of water, then it is made of hydrogen and oxygen.

    (d) If it is made of hydrogen and oxygen, then it is a molecule of water.

27-28. Old Faithful is a geyser located in Yellowstone National Park in Wyoming.

27. If Frank is taking a picture of Old Faithful, then Frank ______________ (choose one: may be, is, is not) in Wyoming.

28. If Frank's cousin Billy is not taking a picture of Old Faithful, then Billy ______________ (choose one: may be, is, is not) in Wyoming.

29. Sabancaya is an active volcano is Peru, and Julio is visiting some family in Peru. Julio ____________ (choose one: may, will, will not) visit Sabancaya.

Part IV – Biconditional Statements and Definitions

We will begin this section with a definition.

A <u>biconditional</u> statement is a statement that has two parts and is only true if both parts are true or both parts are false. Usually, a biconditional statement is written with the words “if and only if” in the middle.

When we want to give a definition of a word in this book, we will generally use biconditional statements.

Example 1: An angle is said to be a right angle *if and only if* its measure is 90º. What does this mean?

This statement means that we can make two statements: (1) If an angle is a right angle, then its measure is 90º, and (2) If the measure of an angle is 90º, then it is a right angle.

Example 2: Consider the statement “$x = 3$ if and only if $x < 7$.” Is this statement true or false?

We will begin by noting that the statement “$x = 3$ if and only if $x < 7$” means two things: (1) If $x = 3$, then $x < 7$, and (2) If $x < 7$, then $x = 3$. In order for the original statement to be true, both of these statements must be true. The first statement is true, but the second is not. Thus, we can say that the original statement is false.

Example 3: Assume that the following statement is true: *Anyone who applies for a job at Nate’s Trucking Company will be hired if and only if he or she has a perfect driving record*. Which of the following statements must be true? (a) Anyone who applies for a job at Nate’s Trucking Company will be hired if he or she has a perfect driving record. (b) Anyone who applies for a job at Nate’s Trucking Company will not be hired if he or she does not have a perfect driving record. (c) Some people who apply for a job at Nate’s Trucking Company and who do not have perfect driving records will be hired. (d) If you apply for a job at Nate’s Trucking Company and are hired, then you have a perfect driving record. (e) If you apply for a job at Nate’s Trucking Company and are not hired, then you do not have a perfect driving record.

We will begin by noting that the original statement says two things: (1) If you apply for a job at Nate’s Trucking Company and have a perfect driving record, then you will be hired, and (2) If you apply for a job at Nate’s Trucking Company and are hired, then you have a perfect driving record.

(a) This statement is equivalent to statement (1) above, and so it must be true.

(b) This statement is equivalent to the contrapositive of statement (2) above, and, since the contrapositive of a statement is always true when the original statement is true, this statement must be true.

(c) This statement is not true because statement (2) above says that you must have a perfect driving record in order to be hired.

(d) This statement is true because it is the same as statement (2) on the previous page.

(e) This statement is equivalent to the contrapositive of statement (1) on the previous page, and, since the contrapositive of a statement is always true when the original statement is true, this statement must be true.

Problems:

1-11. Tell whether each statement below is *true* or *false*.

1. A number is positive if and only if it is greater than -3.

2. $x = y$ if and only if $x + 3 = y + 3$

3. $x^2 = 81$ if and only if $x = 9$

4. An insect is a mosquito if and only if it has wings.

5. $|-5| = 5$ if and only if $3 < 4$

6. $|-5| = 2$ if and only if $7 < 2$

7. In baseball, a team wins a game if and only if that team scores more runs than the other team.

8. $x = 3$ if and only if $x < 4$

9. A figure is a square if and only if it has four sides.

10. Today is Thursday if and only if tomorrow is Friday.

11. An animal is a snake if and only if it is a reptile.

12. Assume that the following statement is true: *Marlene will eat a piece of the pizza if and only if it has Italian sausage on it*. Select all of the statements below that must be true.

    (a) If Marlene eats a piece of the pizza, then the pizza has Italian sausage on it.

    (b) If the pizza has Italian sausage on it, then Marlene will eat a piece of the pizza.

    (c) If Marlene does not eat a piece of the pizza, then the pizza does not have Italian sausage on it.

    (d) If the pizza does not have Italian sausage on it, then Marlene will not eat a piece of it.

    (e) Marlene will eat a piece of the pizza.

    (f) The pizza has Italian sausage on it.

13. Assume that the following statement is true: *$x = 2$ if and only if $x = y$*. Select all of the answers below that must be true.

    (a) $x = 2$

    (b) $x = y$

    (c) $y = 2$

    (d) If $x = 2$, then $x = y$.

    (e) If $x = y$, then $x = 2$.

    (f) If $x \neq y$, then $x \neq 2$.

    (g) If $x \neq 2$, then $x \neq y$.

    (h) If $x < y$, then $x \neq 2$.

    (i) If $x \neq y$, then $x < 2$.

14. Assume that the following statement is true: *Jake will do the dishes if and only if his sister helps him*. Select all of the statements below that must be true.

    (a) Jake will do the dishes.

    (b) Jake will not do the dishes.

    (c) If Jake does the dishes, then his sister will help him.

    (d) If Jake does not do the dishes, then his sister will not help him.

    (e) If his sister helps him, then Jake will do the dishes.

    (f) If his sister does not help him, then Jake will not do the dishes.

15. Assume that the following statement is true: *Two lines are perpendicular if and only if they form right angles*. Select all of the statements below that must be true.

    (a) If two lines are perpendicular, then they form right angles.

    (b) If two lines are not perpendicular, then they do not form right angles.

    (c) If two lines do not form right angles, then they are not perpendicular.

    (d) Two lines are not perpendicular if they form right angles.

    (e) Two lines are not perpendicular if they do not form right angles.

16. Assume that the following statement is true: *A quadrilateral is a parallelogram if and only if its diagonals bisect each other*. Select all of the statements below that must be true.

    (a) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

    (b) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is not a parallelogram.

    (c) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

    (d) The diagonals of a quadrilateral will bisect each other if the quadrilateral is a parallelogram.

    (e) The diagonals of a quadrilateral will not bisect each other if the quadrilateral is not a parallelogram.

17. Assume that the following statement is true: *An object is a diblet if and only if it is jeppable*. Select all of the answers below that must be true.

    (a) All diblets are jeppable.

    (b) Some jeppable objects are not diblets.

    (c) All jeppable objects are diblets.

    (d) If an object is not a diblet, then it is not jeppable.

    (e) If an object is a diblet, then it is jeppable.

    (f) If an object is jeppable, then it might not be a diblet.

    (g) If an object is jeppable, then it is a diblet.

    (h) If an object is not jeppable, then it is not a diblet.

18. Assume that the following statement is true: *In Italian, an object is called a* matita *if and only if it is a pencil*. Select all of the answers below that must be true.

    (a) If an object is a pencil, then it is called a *matita* in Italian.

    (b) If an object is not a pencil, then it is not called a *matita* in Italian.

    (c) There is a pencil somewhere that is not called a *matita* in Italian.

    (d) If an object is called a *matita* in Italian, then it is a pencil.

    (e) If an object is not called a *matita* in Italian, then it is not a pencil.

    (f) There is an object called a *matita* in Italian that is not a pencil.

Part V – Direct and Indirect Proofs

In this section, we will discuss how to write two different kinds of proofs. A proof is a list of statements you know to be true and the reasons why you know these statements are true. In this section, we will only discuss algebraic proofs. The more complicated proofs that deal with geometric concepts will be discussed in later chapters. We will begin by reviewing some properties from Algebra. Also, you may find it helpful to review the sections on linear equations, linear inequalities, and solving systems of equations using substitution in Appendix A before you try to understand the examples in this section.

Let a, b, and c represent real numbers. Then:

- the Addition Property of Equality says that $a = b$ if and only if $a + c = b + c$.
- the Subtraction Property of Equality says that $a = b$ if and only if $a - c = b - c$.
- the Multiplication Property of Equality says that $a = b$ if and only if $ac = bc$ and $c \neq 0$.
- the Division Property of Equality says that $a = b$ if and only if $\frac{a}{c} = \frac{b}{c}$ and $c \neq 0$.
- the Addition Property of Inequality says that $a < b$ if and only if $a + c < b + c$.
- the Subtraction Property of Inequality says that $a < b$ if and only if $a - c < b - c$.
- the Multiplication Property of Inequality says that, if $a < b$ and $c > 0$, then $ac < bc$; if $a < b$ and $c < 0$, then $ac > bc$.
- the Division Property of Inequality says that, if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$; if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.
- the Reflexive Property of Equality says that $a = a$.
- the Symmetric Property of Equality says that $a = b$ if and only if $b = a$.
- the Transitive Property of Equality says that, if $a = b$ and $b = c$, then $a = c$.
- the Substitution Property of Equality says that, if $a = b$, then a may be replaced by b in any equation or inequality.
- the Symmetric Property of Inequality says that $a < b$ if and only if $b > a$.
- the Transitive Property of Inequality says that, if $a < b$ and $b < c$, then $a < c$.
- the Distributive Property says that $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.

Example 1: Suppose that you know that $3x = 24$. Which property allows you to say that $x = 8$?

If we know that $3x = 24$, then we can divide both sides of this equation by 3 to find that $x = 8$. The Division Property of Equality allows us to do this, and so we can say that the Division Property of Equality tells us that $x = 8$.

The next four examples give you some examples of proofs. When you do the practice problems, you do not need to write the third column. We only included the third column so that you could understand what we were doing better.

Example 2: Solve for x in the equation $4x - 5 = 23$, and give a reason for each step. It turns out that $x = 7$, as shown in the proof below.

Statements	Reasons	Explanations
1. $4x - 5 = 23$	1. Given	We were told that $4x - 5 = 23$. When you are told that something is true at the beginning of the problem, you should write "Given" as your reason for knowing this statement.
2. $4x = 28$	2. Addition Property of Equality	We added 5 to both sides of the equation, and the Addition Property of Equality is the property that allows us to do this.
3. $x = 7$	3. Division Property of Equality	We divided both sides of the equation by 4, and the Division Property of Equality is the property that allows us to do this.

Example 3: Given that $4y - 5 < 7(y + 4) - 2$, prove that $y > -\frac{31}{3}$.

There are many different ways to work this problem, and one of them is shown below.

Statements	Reasons	Explanations
1. $4y - 5 < 7(y + 4) - 2$	1. Given	We were told that $4y - 5 < 7(y + 4) - 2$. When you are told that something is true at the beginning of the problem, you should write "Given" as your reason for knowing this statement.
2. $4y - 5 < 7y + 28 - 2$	2. Distributive Property	The Distributive Property tells us that $7(y + 4) = 7y + 28$.
3. $4y - 5 < 7y + 26$	3. Substitution Property of Equality	Note that $28 - 2 = 26$, and so we can substitute 26 for $28 - 2$.
4. $-5 < 3y + 26$	4. Subtraction Property of Inequality	We subtracted 4y from both sides of the inequality, and the Subtraction Property of Inequality is the property that allows us to do this.
5. $-31 < 3y$	5. Subtraction Property of Inequality	We subtracted 26 from both sides of the inequality, and the Subtraction Property of Inequality is the property that allows us to do this.
6. $-\frac{31}{3} < y$	6. Division Property of Inequality	We divided both sides of the inequality by 3, and the Division Property of Inequality is the property that allows us to do this.
7. $y > -\frac{31}{3}$	7. Symmetric Property of Inequality	The Symmetric Property of Inequality states that $a < b$ if and only if $b > a$.

Example 4: Given that $2x + 3y = 7$ and $x - y = 6$, prove that $x = 5$ and $y = -1$.

Once again, there are several ways of solving this problem, and one of them is shown below.

Statements	Reasons	Why did we choose the reason we did?
1. $2x + 3y = 7$ and $x - y = 6$	1. Given	We were told that $x - y = 6$. When you are told that something is true at the beginning of the problem, you should write "Given" as your reason for knowing this statement.
2. $x = 6 + y$	2. Addition Property of Equality	We added y to both sides of the second equation in Statement (1), and the Addition Property of Equality is the property that allows us to do this.
3. $2(6 + y) + 3y = 7$	3. Substitution Property of Equality	We said in Statement (2) that $x = 6 + y$, and so we can substitute $6 + y$ for x in the first equation in Statement (1).
4. $12 + 2y + 3y = 7$	4. Distributive Property	The Distributive Property tells us that $2(6 + y) = 12 + 2y$.
5. $12 + 5y = 7$	5. Distributive Property	When we combine like terms, we are using the Distributive Property. To see why this is true, note that $2y + 3y = y(2 + 3)$, or $5y$. (You could also use the Substitution Property of Equality as your reason here, but the Distributive Property is really a better reason.)
6. $5y = -5$	6. Subtraction Property of Equality	We subtracted 12 from both sides of the equation in Statement (5), and the Subtraction Property of Equality is the property that allows us to do this.
7. $y = -1$	7. Division Property of Equality	We divided both sides of the equation by 5, and the Division Property of Equality is the property that allows us to do this.
9. $x = 6 + (-1)$	9. Substitution Property of Equality	We already said that $y = -1$ and that $x = 6 + y$. So, we can substitute -1 for y in the equation $x = 6 + y$.
10. $x = 5$	10. Substitution Property of Equality	Since $6 + (-1) = 5$, we can substitute 5 for $6 + (-1)$.

All of the proofs that we have done up to this point have been what we call direct proofs. In the next example, we discuss the concept of indirect proofs. To write an indirect proof, you assume that the "given" part is true **and** the negation of the part that you want to prove is true. You work with both of these assumptions, and then you get a contradiction, which is a statement that cannot be true based on your assumptions.

Example 5: Given that 5x + 2 = –18, prove that x must equal –4 using an indirect proof.

Once again, there are several ways to work this problem, and one of them is shown below.

Statements	Reasons	Why did we choose the reason we did?
1. 5x + 2 = –18	1. Given	We were told that 5x + 2 = –18. When you are told that something is true at the beginning of the problem, you should write "Given" as your reason for knowing this statement.
2. x ≠ –4	2. Assumption	We are allowed to assume the negation of the statement that we are trying to prove.
3. 5x ≠ –20	3. Multiplication Property of Inequality	The Multiplication Property of Inequality tells us that we can multiply both sides of Statement (2) by 5.
4. 5x + 2 ≠ –18	4. Addition Property of Inequality	The Addition Property of Inequality tells us that we can add 2 to both sides of Statement (3).
5. x must equal –4	5. Contradiction	Our assumption must be false because Statement (4) is a contradiction to Statement (1).

When you are writing an indirect proof, make sure that you assume the "Given" **and the negation of the statement you are trying to prove**. If you don't do this, your answer will be wrong because you are not using correct logic.

Problems:

1-18. Prove each of the following statements using a direct proof.

1. Given that 5x = 3x – 14, prove that x = –7.

2. Given that $7a - 2 = 3(2a - 4)$, prove that $a = -10$.

3. Given that $5y + 4 = 2 - 3y$, prove that $y = -\frac{1}{4}$.

4. Given that $\frac{p}{5} + 3 = 7$, prove that $p = 20$.

5. Given that $5 - \frac{3w}{4} = 15$, prove that $w = -\frac{40}{3}$.

6. Given that $2k + 3 = 5 - 2(k + 4)$, prove that $k = -\frac{3}{2}$.

7. Given that $2n - 5 = 3 - (n + 4)$, prove that $n = \frac{4}{3}$.

8. Given that $2(d – 3) + 1 = d – 5$, prove that $d = 0$.

9. Given that $3q + 4 \le 5$, prove that $q \le \frac{1}{3}$.

10. Given that $6 – 3m > 4$, prove that $m < \frac{2}{3}$.

11. Given that $x + 4(2x - 1) < 5$, prove that $x < 1$.

12. Given that $5 + 2(3 - c) \leq 4c$, prove that $c \geq \frac{11}{6}$.

13. Given that $3 - \frac{5y}{12} < \frac{1}{4}$, prove that $y > \frac{33}{5}$.

14. Given that $\frac{5}{6}+\frac{2u}{3} > \frac{1}{2}$, prove that $u > -\frac{1}{2}$.

15. Given that $5x + y = 36$ and $x – 5y = 2$, prove that $x = 7$ and $y = 1$.

16. Given that $2x + 3y = –17$ and $x – 4y = 8$, prove that $x = –4$ and $y = –3$.

17. Given that $2x + y = -4$ and $3x - y = -11$, prove that $x = -3$ and $y = 2$.

18. Given that $8x - y = 4$ and $5x + 2y = -8$, prove that $x = 0$ and $y = -4$.

19-25. Prove each of the following statements using an indirect proof.

19. Given that $5a - 4 = 31$, prove that $a = 7$.

20. Given that $2(b + 5) = 12$, prove that $b = 1$.

21. Given that $\frac{5c}{8} + 1 = -9$, prove that $c = -16$.

22. Given that $\frac{2x}{3} - 4 = 1$, prove that $x = \frac{15}{2}$.

23. Given that 2 = 3 + 7y, prove that $y = -\frac{1}{7}$.

24. Given that 3z + 1 < 5, prove that $z < \frac{4}{3}$.

25. Given that 3 – 2n ≥ 5, prove that n ≤ –1.